

# Antistrong Digraphs, Shandong course 2026

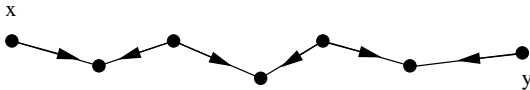
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- In a *digraph*  $D$ , an **antidirected path** is a path in which the arcs alternate and beginning and ending with a forward arc.

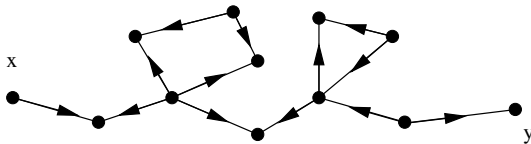


- Motivation:** find similar (algorithmic) results between directed paths and antidirected paths, but...

## Theorem 1 (A. Yeo, 2014)

Given two vertices  $x$  and  $y$  of  $D$ , it is NP-complete to decide if  $D$  admits an antidirected path from  $x$  to  $y$ .

- A **forward antidirected trail** is a trail (no repeated arc) in which the arcs alternate and beginning and ending with a forward arc.

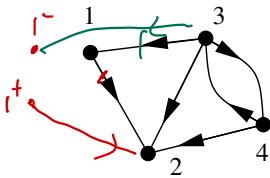


- A **forward antidirected trail** is a trail (no repeated arc) in which the arcs alternate and beginning and ending with a forward arc.

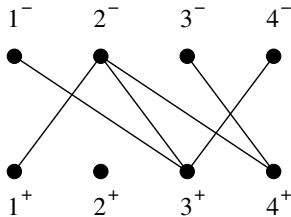
## Theorem 2

*It is polynomial to check if there exists a forward antidirected trail from  $x$  to  $y$ .*

*Proof* :  $B(D)$ : the (oriented) adjacency bipartite representation of  $D$ .



D



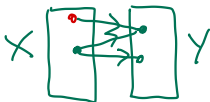
B(D)

- A digraph is **antistrong** if there exists a forward andirected trail from  $x$  to  $y$  for all  $x, y \in V(D)$ .

## Theorem 3

*For  $|D| \geq 3$ ,  $D$  is antistrong iff  $B(D)$  is connected.*

- We provide some algorithmic results related to antistrong connectivity.
- First an easy one: **in polytime we can check antistrong connectivity.**

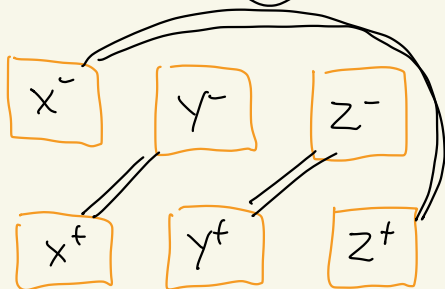
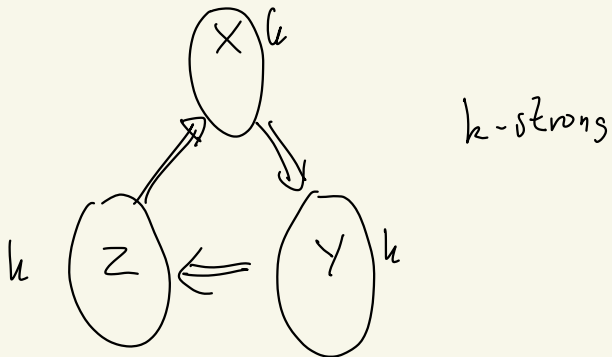


### Proposition 4

*No bipartite digraph is antistrong.*

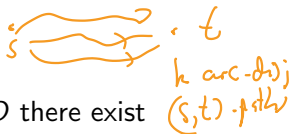
A digraph is  **$k$ -strong** if it has at least  $k + 1$  vertices and it remains strong after deletion of any set of at most  $k - 1$  vertices. The digraph obtained from three disjoint independent sets  $X, Y, Z$  each of size  $k$  by adding all arcs from  $X$  to  $Y$ , from  $Y$  to  $Z$ , and from  $Z$  to  $X$  is  $k$ -strong. However,  $B(D)$  has three connected components.

This shows that no condition on the strong connectivity will guarantee that a digraph is antistrong.



# Direct results: $k$ -antistrong digraph

$k$ -arc-strong:



- $D$  is  **$k$ -arc-antistrong** if for every  $x, y \in D$  there exist  $k$ -arc-disjoint antirected trails from  $x$  to  $y$ .

## Theorem 5

$D$  is  $k$ -arc-antistrong iff  $B(D)$  is  $k$ -edge-connected.

Corollaries:

- In polytime we can check  $k$ -arc-antistrong connectivity.

0

0

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## Theorem 6

*If  $D$  is  $2k$ -arc-antistrong, then it contains  $k$  arc-disjoint antistrong spanning subdigraphs.*

**Proof:** Since  $D$  is  $2k$ -arc-antistrong,  $B(D)$  is  $2k$ -edge-connected. We can now use Nash-Williams' theorem on edge-disjoint spanning tree to deduce that  $B(D)$  has  $k$  edge-disjoint spanning trees. Translating each of these trees in  $B(D)$  back to arc-sets in  $D$  now gives the required set of  $k$  arc-disjoint antistrong spanning subdigraphs of  $D$ . □

The existence of  $k$ -edge-disjoint spanning trees in a graph  $G$  can be checked in polynomial time using Edmonds' algorithm for matroid partition:  $G$  has  $k$  edge-disjoint spanning trees if and only if the matroid union of  $k$  copies of the circuit matroid  $M(G)$  has  $k$  edge-disjoint bases.

### Theorem 7

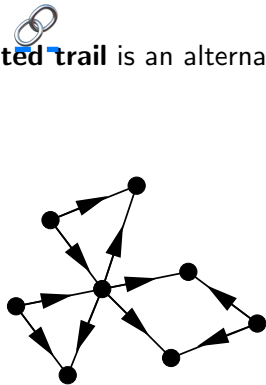
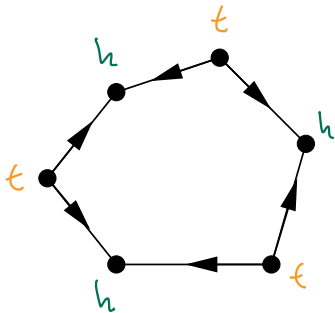
*There exists a polynomial time algorithm which for a given digraph  $D$  and a natural number  $k$  either returns  $k$  arc-disjoint spanning antistrong subdigraphs  $D_i = (V, A_i)$ ,  $i \in [k]$  of  $D$  or correctly answers that no such set exists.*

### Theorem 8 (Yeo, 2009)

*It is NP-complete to decide whether a digraph  $D$  contains two spanning strong subdigraphs  $D_1, D_2$  which are arc-disjoint.*


# Direct results: a matroid for antistrong connectivity

- A **CAT** or **closed antidirected trail** is an alternating closed trail.



- The cat-free sets of arcs of  $D$  form a matroid on the arcs of  $D$ .


# A matroid for antistrong connectivity

Recall from matroid theory that  $B(D)$  is connected if and only if the cycle matroid  $M(B(D))$  has rank  $|V(B(D))| - 1$  

For  $F \subseteq A$ , we denote by  $h(F)$  and  $t(F)$  the numbers of vertices that are heads, respectively tails, of one or more arcs in  $F$ .

Recall that the independent sets of the cycle matroid  $M(G)$  of a graph  $G = (V, E)$  are those subsets  $I \subseteq E$  for which we have  $|I'| \leq \nu(I') - 1$  for all  $\emptyset \neq I' \subseteq I$ , where  $\nu(I')$  is the number of end vertices of the edges in  $I'$ . we define set  $I$  of arcs in a digraph

$D = (V, A)$  to be **independent** if

$I'$  

$$|I'| \leq h(I') + t(I') - 1 \quad \text{for all } \emptyset \neq I' \subseteq I, \quad (1)$$

A set  $S \subseteq A$  is **dependent** if it is not independent.

## Proposition 9

Let  $D = (V, A)$  be a digraph. A subset  $I \subseteq A$  is independent if and only if the corresponding edge set  $I$  in  $B(D)$  forms a forest. Every inclusion-minimal dependent set  $S \subseteq A$  corresponds to a cycle in  $B(D)$  and conversely.

Denote this matroid  $M_{AS}(D)$

- Note that  $M_{AS}(D)$  has rank  $2n-1$  if and only if  $B(D)$  is connected
- Independent sets in  $M_{AS}(D)$  are the CAT-free subsets of  $A(D)$

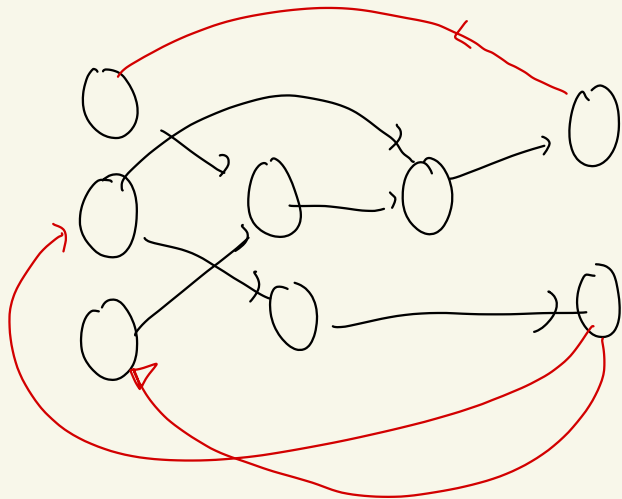
## Theorem 10

*There exists a polynomial time algorithm for finding, for a given digraph  $D = (V, A)$  on at least 3 vertices, a minimum cardinality set of new arcs  $F$  such that the digraph  $D' = (V, A \cup F)$  is antistrong.*

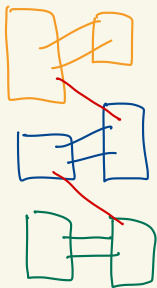
The complexity of the analogous question for  $k$ -arc-antistrong connectivity is open:

## Problem 11

*Given a digraph  $D$  and a natural number  $k$ , can we find in polynomial time a minimum cardinality set of new arcs whose addition to  $D$  results in a digraph  $D'$  which is  $k$ -arc-antistrong?*



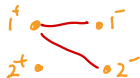
## proof of Theorem 10



Same as minimum #edges  $A'$  to add s.t.  
 $B(D+A')$  is connected

As  $n \geq 3$   $B(D)$  has  $\geq 6$  vertices so we  
can avoid edges of the kind  $v' \rightarrow v''$

Now we add all missing edges except  $v' \rightarrow v''$   
to  $B(D)$  of weight 1 and original weight 0  
solve Minimum sp. tree problem



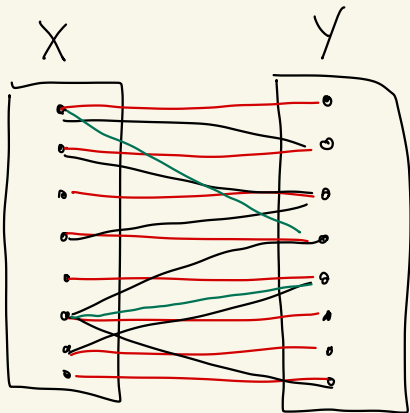
Problem 11 is easily seen to be equivalent to the following problem on edge-connectivity augmentation of bipartite graphs.

### Problem 12

*Given a natural number  $k$  and a bipartite graph  $B = (X, Y, E)$  with  $|X| = |Y| = p$  which admits a perfect matching  $M$  in its bipartite complement, find a minimum cardinality set of new edges  $F$  such that  $F \cap M = \emptyset$  and  $B + F$  is  $k$ -edge-connected and bipartite with the same bipartition as  $B$ .*

$$M = \text{---}$$

$$E = \text{---}$$



## CAT-free orientation:

### Theorem 13

Let  $G = (V, E)$  with  $|E| \leq 2|V| - 1$ .

*$G$  has a CAT-free orientation iff:*

$|E(H)| \leq 2|V(H)| - 1$  for all  $(\neq \emptyset)$  subgraphs  $H$  of  $G$  ~~(\*)~~

$|E(H)| \leq 2|V(H)| - 2$  for all  $(\neq \emptyset)$  bip. subgraphs  $H$  of  $G$  ~~(\*\*)~~

Remarks:

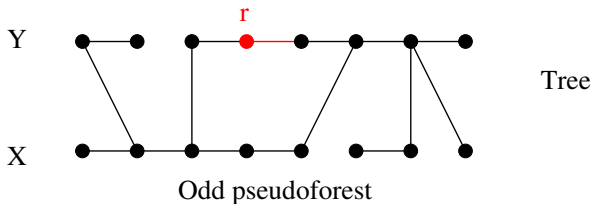
- ~~(\*)~~ and ~~(\*\*)~~ are necessary.
- No bipartite digraph is antistrong.

**Proof:** In two steps:

- A graph is an **odd-pseudoforest** if each of its connected components contains at most one cycle which is odd if it exists.
- **Claim 1:**  $G$  satisfies  $(*)$  and  $(**)$  iff it can be (edge)-partitioned into a forest and an odd pseudoforest.
- **Claim 2:** Every graph which is the (edge)-union of a forest and an odd pseudoforest admits a cat-free orientation.

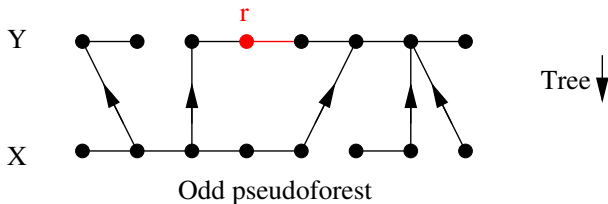
- **Claim 2:** Every graph which is the (edge)-union of a forest and an odd pseudoforest admits a cat-free orientation.

**Proof:**



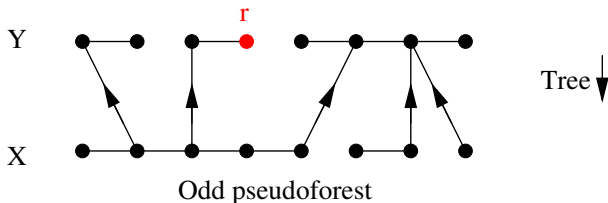
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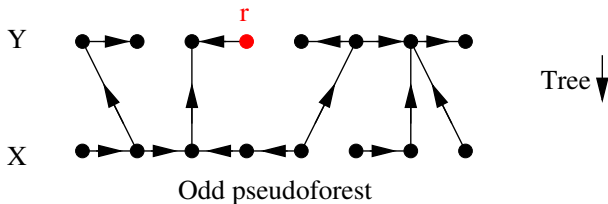
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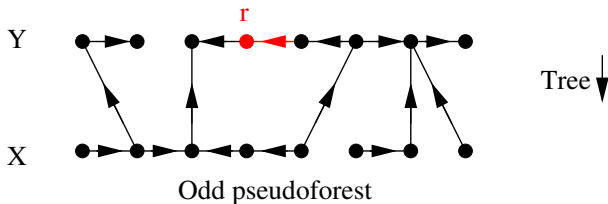
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**Proof:**



- **Claim 2:** Every graph which is the (edge)-union of a forest and an odd pseudoforest admits a cat-free orientation.

**Proof:**



- **Claim 1:**  $G$  satisfies

$$|E(H)| \leq 2|V(H)| - 1 \quad \text{for all } (\neq \emptyset) \text{ subgraphs } H \text{ of } G \quad (**)$$

$$|E(H)| \leq 2|V(H)| - 2 \quad \text{for all } (\neq \emptyset) \text{ bip. subgraphs } H \text{ of } G \quad (***)$$

iff it can be (edge)-partitioned into a forest and an odd pseudoforest.

**Proof:** *graph theory vs matroids*

- Let  $E$  be a set and  $f : 2^E \rightarrow \mathbb{Z}$  a submodular, nondecreasing function which is nonnegative on  $2^E \setminus \{\emptyset\}$ .

### Theorem 14 (J. Edmonds, 1970)

The sets  $I \subseteq E$  s.t.  $\forall \emptyset \neq I' \subseteq I \quad |I'| \leq f(I')$  form a matroid  $M_f$  on  $E$ .

The rank of a subset  $S \subseteq E$  in  $M_f$  is given by the min-max formula:

$$r_f(S) = \min_{(S_0, S_1, \dots, S_k)} \left\{ |S_0| + \sum_{i=1}^k f(S_i) \right\}$$

where the min is taken over all partitions  $(S_0, S_1, \dots, S_k)$  of  $S$ .

- **Claim 1:**  $G$  satisfies

$$|E(H)| \leq 2|V(H)| - 1 \quad \text{for all } (\neq \emptyset) \text{ subgraphs } H \text{ of } G \quad (*)$$

$$|E(H)| \leq 2|V(H)| - 2 \quad \text{for all } (\neq \emptyset) \text{ bip. subgraphs } H \text{ of } G \quad (**)$$

iff it can be (edge)-partitioned into a forest and an odd pseudoforest.

### Proof:

- **Ex1:**  $f = \nu - 1$ , where  $\nu(I)$  is the nb of vertices incident with edges in  $I$ , **the cycle matroid**.
- **Ex2:**  $f = \nu - \beta$ , where  $\beta(I)$  is the nb of bipartite components formed by the edges of  $I$ , **the even bicircular matroid**.
- **Ex3:**  $f = 2\nu - 1 - \beta$ . Independent in  $M_f$  iff satisfies (1) and (2).

- **Claim 1:**  $G$  satisfies

$$|E(H)| \leq 2|V(H)| - 1 \quad \text{for all } (\neq \emptyset) \text{ subgraphs } H \text{ of } G \quad (*)$$

$$|E(H)| \leq 2|V(H)| - 2 \quad \text{for all } (\neq \emptyset) \text{ bip. subgraphs } H \text{ of } G \quad (**)$$

iff it can be (edge)-partitioned into a forest and an odd pseudoforest.

**Proof:**

- Every independent set of  $M_f \vee M_g$  is an independent set of  $M_{f+g}$ , but in general the converse is not true.

**Theorem 15** (N. Katoh and S. Tanigawa, 2012)

$M_f \vee M_g = M_{f+g}$  if for every  $S \subset E$  the min in the ranks  $r_f(S)$  and  $r_g(S)$  is attained for the same partition of  $S$ .

(recall  $r_f(S) = \min_{(S_0, S_1, \dots, S_k)} \left\{ |S_0| + \sum_{i=1}^k f(S_i) \right\}$ )

- **Corollary:**  $M_{2\nu-1-\beta} = M_{\nu-1} \vee M_{\nu-\beta}$  and Claim 1 is proven.

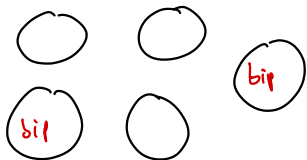
In general, for graphs:

### Theorem 16

*A graph  $G = (V, E)$  has an antistrong orientation if and only if*

$$e(Q) \geq |Q| - 1 + b(Q) \quad (4)$$

*for all partitions  $Q$  of  $V$ , where  $e(Q)$  denotes the number of edges of  $G$  between the different parts of  $Q$  and  $b(Q)$  the number of parts of  $Q$  which induce bipartite subgraphs of  $G$ .*

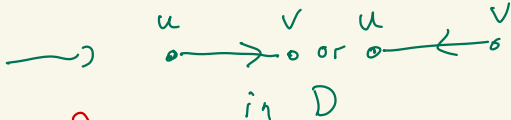


# Antistrows orientation, via matrix intersection

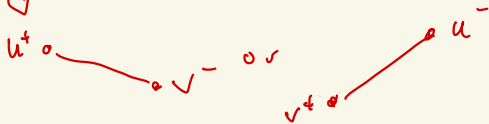
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in  $G$



in  $D$

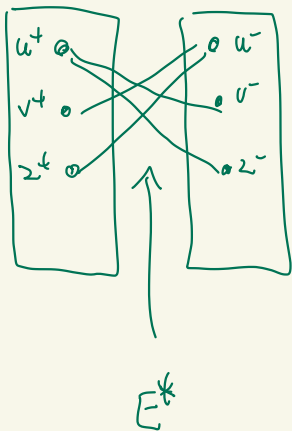


in  $B(D)$

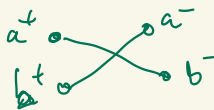
$$E^* = \{ u^+v^-, v^+u^- \mid uv \in E \}$$

$M_1$ :  $E' \subseteq E^*$  independent iff no  $u^+v^-, v^+u^- \in E'$

$M_2$ :  $E'' \subseteq E^*$  independent iff no cycle.



Circuit of  $M_C$ ?



$G$  has an antistrong orientation



$M_1$  and  $M_2$  have a common independent set of size  $2n-1$

- We can decide if a graph admits an antistrong orientation in polytime.
- Every 4-edge-connected nonbipartite graph has an antistrong orientation.
- Every nonbipartite graph with three edge disjoint spanning trees has an antistrong orientation.

Every 4-edge connected non-bipartite graph has  
an antistrong orientation

P: we need to show that  $e(Q) \geq |Q| - 1 + b(Q)$

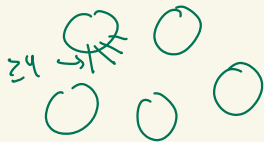
if  $|Q| = 1$  (i.e.  $Q = \{v\}$ ) then

$e(Q) = 0 = 1 - 1 + 0$  as  $G$  is not bipartite

if  $|Q| > 1$  then  $e(Q) \geq 2|Q|$  as  $G$  is 4-edge connected

$$> |Q| - 1 + |Q|$$

$$\geq |Q| - 1 + b(Q)$$



Every non-bipartite graph  $G$  with 3 edge-disjoint sp. trees  
has an antisymmetric orientation

$P$ : Consider a partition  $\mathcal{Q} = V_1 \cup V_2 \cup \dots \cup V_p$  of  $V$

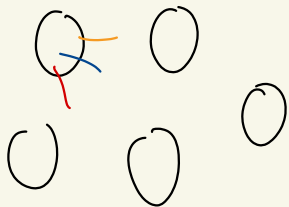
if  $p=1$ , then  $e(\mathcal{Q})=0=1-1+0$  as  $G$  is not bipartite

if  $p \geq 1$  then  $e(\mathcal{Q}) \geq 3(p-1)$

$$= p-1 + 2(p-1)$$

$$\geq p-1 + p \quad \text{as } p \geq 2$$

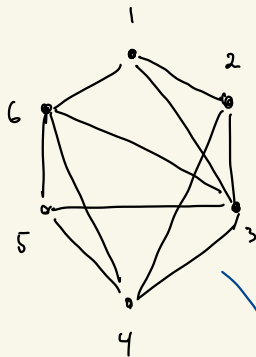
$$\geq p-1 + \delta(\mathcal{Q})$$



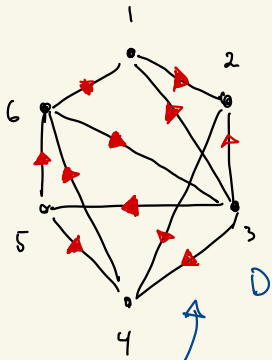
A **2-detachment** of a graph  $G = (V, E)$  is any graph  $H = (V' \cup V'', E')$  which can be obtained from  $G$  by replacing every vertex  $v \in V$  with two new vertices  $v', v''$  and then for each original edge  $uv$  adding precisely one of the four edges  $u'v', u'v'', u''v', u''v''$  to  $E'$ .

## Lemma 17

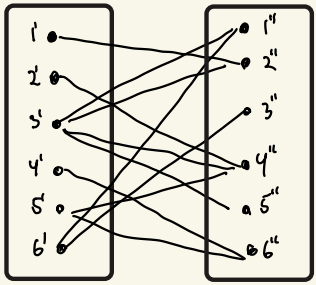
*A graph  $G = (V, E)$  has an antistrong orientation if and only if  $G$  has a 2-detachment  $H = (V' \cup V'', E')$  which is connected and bipartite with bipartition  $V', V''$  (we call such a 2-detachment **good**).*



antistong  
 orientation



Good  
 2-detachment



$B(D)$

We can now use Theorem 16 to deduce the following:

### Theorem 18

*A graph  $G = (V, E)$  has a good 2-detachment if and only if*

$$e(Q) \geq |Q| - 1 + b(Q) \quad (5)$$

*for all partitions  $Q$  of  $V$ . Furthermore, there exists a polynomial time algorithm which returns such a 2-detachment when it exists and otherwise returns a certificate, in terms of a partition violating (5), that no such detachment exists.*

## Theorem 19 (Bang-Jensen and Yeo, 2010)

*It is NP-complete to decide whether a given digraph  $D$  contains a spanning strong subdigraph  $H$  such that  $UG(D - A(H))$  is connected.*

If we replace “strong” by “antistrong” above, the problem becomes solvable in polynomial time.

## Theorem 20



*We can decide in polynomial time for a given digraph  $D = (V, A)$  on  $n$  vertices whether  $D$  contains a spanning antistrong subdigraph  $H = (V, A')$  such that  $UG(D - A')$  is connected.*

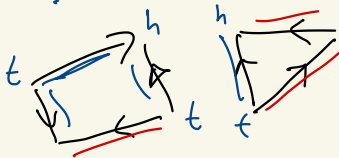
proof of Theorem 20

$D$  has a spanning acyclic subdigraph  $H=(V, A')$  s.t  
 $UG(D-A')$  is connected if and only if

The matroid  $M_1 \vee M_2$  has an independent set of size  
 $2n-1 + n-1 = 3n-2$ , when  $M_1$  is the cycle matroid of  $UG(D)$   
 and  $M_2 = M_{As}(D)$

We can find a largest  $A' \subseteq A$  so that  $A' = A_1 \cup A_2$   
 where  $A_1$  is acyclic in  $UG(D)$  and  $A_2$  is CAT-free (acyclic in  
 $B(D)$ )  
 in polynomial time because  $M_1 \vee M_2$  is a  
 partition matroid.

$M_1$       $M_2$   
     



## Question 21

*Can we decide in polynomial time whether  $D$  contains arc-disjoint spanning subdigraphs  $D_1, D_2$  such that  $D_1$  is antistrong and  $D_2$  is strongly connected?*

Inspired by Theorem 20 it is natural to ask the following intermediate question.

## Question 22

*Can we decide in polynomial time whether  $D$  contains arc-disjoint spanning subdigraphs  $D_1, D_2$  such that  $D_1$  is antistrong and  $UG(D_2)$  is 2-edge-connected?*

## Conjecture 23

*There exists a natural number  $k$  such that every digraph  $D$  which is both  $k$ -arc-strong and  $k$ -arc-antistrong has arc-disjoint strong spanning subdigraphs  $D_1, D_2$ .*

### Problem 24

*Suppose  $G$  satisfies (\*) and (\*\*) so that it has an edge-partition into a spanning tree  $T$  and an odd pseudo forest  $P$ . Can we find such a partition which minimizes the number of connected components in  $P$  in polynomial time?*

### Problem 25

*Can we decide in polytime if  $G$  has an orientation which is strong and antistrong?*

1 •  
2 ✖  
3 •  
4 •  
5 ✖  
6 •  
7 •

• 1  
✖ 2  
• 3  
• 4  
✖ 5  
• 6  
• 7